Indian Statistical Institute
B.Math. (Hons.) I Year
First Semester Examination, 2005-2006
Probability Theory-I

Time: 3 hrs   Date: 21-11-05   Max. Marks : 100

Note: The paper carries 110 marks. Any score above 100 will be treated as 100.

1. A card from a standard pack of 52 cards is lost. From the remaining cards of the pack, 2 cards are drawn at random and both found to be spades. What is the probability that the missing card is also a spade? [10]

2. (i) Let $0 < p < 1$. Let f be defined on $\mathbb{R}^2$ by

$$f(i,\ j) = \begin{cases} p^2(1-p)^{j-2}, & \text{if } i = 1, \ldots, j-1 \text{ and } j = 2, 3, \ldots \\ 0 \qquad\qquad , & \text{otherwise} \end{cases}$$

Show that f is a 2-dimensional discrete density function.

(ii) Let (X, Y) be a 2-dimensional discrete random variable with f in (i) above as its discrete density function. Find the marginal discrete density functions. Are X and Y independent?

(iii) Describe (X, Y) in terms of a success-failure experiment. [9+7+4]

3. (a) Let X, Y be independent discrete random variables having finite expectations. Show that the random variable XY also has finite expectation and that $E(X\ Y) = E(X) \cdot E(Y)$.

b) Let X, Y be discrete random variables such that $E(X\ Y) = E(X) \cdot E(Y)$. Are X and Y necessarily independent?

(c) Does existence of $E(X)$ and $E(Y)$ imply the existence of $E(X\ Y)$ in general? [7+6+7]

4. Let X, Y be independent random variables having Poisson distribution with parameters $\lambda_1 > 0,\ \lambda_2 > 0$ respectively.

   (i) Find the distribution of $X + Y$.

   (ii) Find $E(Y/X + Y = k)$ for $k = 0, 1, , 2, \ldots$ [8+7]

5. Let X have $N(\mu,\ \sigma^2)$ distribution. Find the probability density function of $Y = e^X$. [10]

6. Let $f(x) = Ce^{-|x|}, \ -\infty < x < \infty$.

   (i) Find C so that $f(\cdot)$ is a probability density function on $I\!R$.

   (ii) Let X be an absolutely continuous random variable having $f(\cdot)$ above as its probability density function. Find the moment generating function $m_X(t)$ of X, indicating for what t it exists. [9 +11]

7. Let X be an absolutely continuous nonnegative random variable such that

$$P(X \geq x) = \frac{1}{(1+x)^\alpha}\ , \quad x \geq 0$$

   where $\alpha > 0$ is a constant.

   (a) Find the probability density function of X.

   (b) Show that $E(X)$ exists if and only if $\alpha > 1$. [7+8]